stochastic
processes
and their

www.elsevier.com/locate/spa

A one-dimensional Poisson growth model with non-overlapping intervals

D.J. Daley[a,*], C.L. Mallows[b], L.A. Shepp[c]

[a] School of Mathematical Sciences, Australian National University, Canberra, ACT 0200, Australia
[b] AT & T Labs, 180 Park Avenue, Florham Park, NJ, 07932-0971, USA
[c] Statistics Department, Rutgers University, Hill Center, New Brunswick, NJ, 08855-1179, USA

Received 2 November 1998; received in revised form 30 August 1999; accepted 19 May 2000

Abstract

Suppose given a realization of a Poisson process on the line: call the points 'germs' because at a given instant 'grains' start growing around every germ, stopping for any particular grain when it touches another grain. When all growth stops a fraction e^{-1} of the line remains uncovered. Let n germs be thrown uniformly and independently onto the circumference of a circle, and let grains grow under a similar protocol. Then the expected fraction of the circle remaining uncovered is the nth partial sum of the usual series for e^{-1}. These results, which sharpen inequalities obtained earlier, have one-sided analogues: the grains on the positive axis alone do not cover the origin with probability $e^{-1/2}$, and the conditional probability that the origin is uncovered by these positive grains, given that the germs n and $n+1$ coincide, is the nth partial sum of the series for $e^{-1/2}$. Despite the close similarity of these results to the rencontre, or matching, problem, we have no inclusion–exclusion derivation of them. We give explicitly the distributions for the length of a contiguous block of grains and the number of grains in such a block, and for the length of a grain. The points of the line not covered by any grain constitute a Kingman-type regenerative phenomenon for which the associated p-function $p(t)$ gives the conditional probability that a point at distance t from an uncovered point is also uncovered. These functions enable us to identify a continuous-time Markov chain on the integers for which $p(t)$ is a diagonal transition probability.

MSC: primary 60D05; secondary 62M30; 60G55

1. Introduction: The model

Häggström and Meester (1996) describe the following germ–grain model for a d-dimensional (d-D) random set with generic realization Ξ, calling it the *dynamic lilypond model*. The *germs* are the points $\{P_i\}$ of a stationary Poisson process at unit rate in d-D euclidean space. At time $t=0$ spheres start growing around each and every germ at the same rate (without loss of generality we assume unit rate), such growth stopping for any sphere $\mathscr{S}$ when it touches another sphere. The spheres so formed are

* Correspondence address. Centre for Mathematical Sciences, Munich University of Technology, 80290 Munich, Germany. Tel.: +49-89-28928617; fax: +49-89-28928464.

E-mail address: daryl.daley@anu.edu.au (D.J. Daley).

0304-4149/00/$ - see front matter
PII: S0304-4149(00)00049-1

the *grains* of a germ–grain model for Ξ, the union of all the grains. They show that, for any finite dimension d, the model is well-defined and prove that with probability one there is no infinite cluster of contiguous grains.

Independently, Daley, et al. (1999) (here after [DSS]) discuss the same model, calling it the PMS model, because it has Poisson-distributed germs and Maximally non-overlapping Spherical grains. Their prime concern is to find the volume fraction ϖ_d for such random sets, i.e. the proportion of space covered by Ξ. They simulated the PMS model in dimensions $d=1$, 2 and 3 to estimate both ϖ_d and the distribution of the volume of the grains, exploiting the germs as points of a marked point process with the grains as marks. They obtained upper and lower bounds on these distributions and the ϖ_d; in the 1-D case their bounds are within about 1% of the true value.

In this paper we consider the 1-D PMS model and establish detailed analytic results, some of them having a beautifully simple form. In particular, we show that the complement $\mathscr{U}$ of Ξ, i.e. that part of space which remains uncovered, constitutes a regenerative phenomenon in the sense of Kingman (1972): indeed, regarding grains as closed sets, Ξ consists of alternating open and closed intervals, of which the lengths are independent and follow, respectively, an exponential distribution and a mixture of convolutions of the same exponential distribution (the tail of this mixture is expressible in terms of a modified Bessel function). We also find the distributions of the number of germs in a closed interval and the lengths of a typical grain and of a typical grain at the end of a closed interval.

Knowing the joint probability that points distance t apart are both covered (or, both uncovered) means that we know the product second moment of the indicator function of the stationary random set process.

The PMS model is the natural completion of a basic model introduced by Stienen (1982) and discussed in Stoyan et al. (1995, p. 218); its second moment properties are described in Schlather and Stoyan (1997). Some properties of the 1-D PMS model are simpler both to describe and to derive than for the Stienen model.

2. The regenerative nature of the 1-D PMS germ–grain model

While the major interest in applications of germ–grain models of random sets is indeed that part of space that is *covered* (i.e. Ξ), the key to our analysis of the 1-D model is its complement $\mathscr{U}$ (Fig. 1). (For definiteness, we assume that the grains, i.e. intervals in 1-D, are *closed* intervals so that touching grains have an end-point in common.) To see this, observe that any point $t \in \mathscr{U}$ is not covered because the grains growing around the points nearest to t on both the right and left, at distances X_t^+ and X_t^- from t say, have grown to 'radii' (i.e. half-interval length of the grain) R_t^+ and R_t^- say, satisfying $R_t^+ < X_t^+$, and $R_t^- < X_t^-$, respectively. Denoting the distances $Y_t^+ = X_t^+ - R_t^+$ and $Y_t^- = X_t^- - R_t^-$ of t to the nearest points of $\mathscr{U}^c$, i.e. of Ξ, it follows from the basic lack of memory property of a Poisson process that these distances Y_t^+ and Y_t^- are independent exponential random variables, with the same distribution as the distances between consecutive points of the Poisson process. *This independence property is crucial to our analysis.*

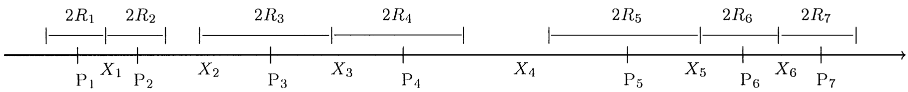

Fig. 1. Germs P_i, grains of length $2R_i$, and intervals of length X_i between germs P_i and P_{i+1} for the 1-D germ–grain model.

The lack of memory property of the points of $\mathscr{U}$ implies that the alternating subintervals that make up $\mathscr{U}$ and its complement, are those of an alternating renewal process, with the uncovered intervals exponentially distributed. We find the distribution of the covered intervals in Section 5.

3. The coverage of a randomly chosen point of $\mathbb{R}$

For the simulation methods used in [DSS] to estimate the volume fraction ϖ_d, it is essential to be able to use a finite number of steps to determine the radius R^* say of a given grain in terms of the relative positions of the germs. An algorithm given there simplifies very much in the 1-D case for which

$$R^* = \min(R'_+, R''_+), \tag{3.1}$$

where the r.v.s R'_+ and R''_+ are independent, and identically distributed like R_+ say. Here R_+ is the radius of a grain centred on the germ P_1 say, where the only germs are $P_1, P_2, \ldots$, the successive points of a Poisson process at unit rate on the positive half-line (Section 6 of [DSS]). To check (3.1), suppose given a doubly infinite sequence of germs $\ldots, P_{-1}, P_0, P_1, \ldots$. The radius R'_+ of the grain with germ P_0 *if only the germs to its right* influence the radius, would be distributed like R_+. Similarly, the radius R''_+ of the grain with germ P_0 if only the germs to its left influence the radius, would again be distributed like R_+, and because of the independence of germs to the right and left of an arbitrarily chosen germ, R'_+ and R''_+ are independent. Moreover, (3.1) clearly holds as the radius of the grain when the influence of points to both the left and right are considered.

Theorem 1. *When grains grow around the germs* $P_1, P_2, \ldots$, *located at the points of a Poisson process at unit rate on the positive half-line and there are no other germs, the radius* R_+ *of the left-most grain has distribution*

$$\Pr\{R_+ > x\} = \mathrm{e}^{-x} \exp[\tfrac{1}{2}(\mathrm{e}^{-2x} - 1)]. \tag{3.2}$$

Proof. We introduce two functions that take note of the temporal development of the grains, using the scenario of germs as in the Theorem. We note that P_1 is at a distance X_0 from the origin O, with X_0 following a unit exponential distribution, and the locations of P_i $(i = 2, 3, \ldots)$ relative to P_1 are independent of X_0. We first find

$$q(y) = \Pr\{\text{grain growing around } P_1 \text{ has covered O by time } y\}. \tag{3.3}$$

The event here occurs only if P_1 is located at some distance x from O with $x < y$, and, given x, the interval $(x, 2x)$ has no germs and the point $2x$ would not have been covered at time x by grains growing to its right. These three events have respective probabilities $e^{-x}\,dx$, e^{-x} and $1-q(x)$, respectively, so

$$q(y)=\int_0^y e^{-x}e^{-x}[1-q(x)]\,dx. \tag{3.4}$$

Differentiating,

$$\frac{q'(y)}{1-q(y)}=e^{-2y}.$$

Integrating and using $q(0+)=0$, gives $q(y)$ explicitly (we state it formally for later use). □

Lemma 2.

$$1-q(y)=\exp[-\tfrac{1}{2}(1-e^{-2y})]. \tag{3.5}$$

Now consider $\Pr\{R_+>x\}$. For R_+ to exceed x, the grain centred on P_1 must still be growing at time x, which is possible only if there are no germs in the interval (P_1, P_1+x), and if at time x the point P_1+x has not been covered by grains growing around the germs $P_2, P_3, \ldots$. These events have probabilities e^{-x} and $1-q(x)$, respectively, and are independent, so their product equals $\Pr\{R_+>x\}$, and is as given at (3.2).

Combining Eqs. (3.1) and (3.2) shows that $\Pr\{R^*>x\}=[\Pr\{R_+>x\}]^2=e^{-2x}\exp(e^{-2x}-1)$, so a typical grain has length (i.e. 1-D volume) $V=2R^*$ whose distribution is given in the next result.

Theorem 3. *The length V of the grain grown around a randomly chosen germ has*

$$\Pr\{V>y\}=e^{-y}\exp(e^{-y}-1). \tag{3.6}$$

Since the density of germs is one, the volume fraction is just

$$\varpi_1=E(V)=\int_0^\infty e^{-y}\exp(e^{-y}-1)\,dy=\int_0^1 e^{u-1}\,du=1-e^{-1}. \tag{3.7}$$

Now here is another derivation of (3.7), more probabilistic in flavour. The volume fraction ϖ_1 equals the probability that a randomly chosen point in $\mathbb{R}$, t say, is covered. The complementary event, of t being uncovered, can occur only if the grains grown around the germs on each side of t, in the absence of germs on the other, do not ultimately reach and cover t. From the independence properties of the Poisson germs, these two events are independent, and have the same probability Q say, and $Q=1-q(\infty)=e^{-1/2}$. Then $Q^2=e^{-1}$, the complement of which is as at (3.7).

Theorem 4. (a) *The probability ϖ_1 that a randomly chosen point of $\mathbb{R}$ is covered by a grain equals* $1-e^{-1}$.

(b) *In the setting of Theorem* 1, *the probability that the origin is not covered by the grain with centre* P_1 *is* $Q=1/\sqrt{e}$.

Remark 3.1. In the setting preceding (3.3), we can also evaluate Q by

$$e^{-1/2} = Q = \Pr\{X_0 > R_+\} = E(I\{X_0 > R_+\}) = E(\Pr\{X_0 > R_+|R_+\}) = E(e^{-R_+}). \tag{3.8}$$

Remark 3.2. Numerical evaluation gives $\varpi_1 = 1 - 1/e = 0.6321205$, consistent with the estimate 0.6322 with standard deviation 0.00024 obtained in the simulations reported in [DSS].

4. The uncovered set $\mathscr{U}$ as a regenerative set

We have already remarked in Section 2 that the uncovered intervals $\mathscr{U}$ of a realization of the 1-D PMS model constitute the regeneration sets of a standard regenerative process. Every such process has a Kingman p-function satisfying $p(t) \to 1$ $(t \downarrow 0)$ and, given any positive integer n and any ascending sequence $0 = t_0 < t_1 < \cdots < t_n$ of points on the non-negative half-line,

$$\Pr\{t_i \in \mathscr{U}(i = 1,\dots,n) \mid t_0 \in \mathscr{U}\} = \prod_{i=1}^{n} p(t_i - t_{i-1}). \tag{4.1}$$

Such a standard p-function is characterised by the measure μ in the representation

$$\int_0^\infty e^{-\theta t} p(t)\,dt = \frac{1}{\theta + \int_{(0,\infty]}(1 - e^{-\theta x})\mu(dx)},$$

where the denominator is finite for $\mathrm{Re}(\theta) > 0$ and μ can be interpreted as a multiple of a lifetime distribution for one of the two components of an alternating renewal process in which the other is an exponential lifetime distribution with mean $1/\mu((0,\infty])$ (Kingman, 1964; see e.g. Kingman, 1972). The diagonal transition probability functions of continuous-time Markov chains on a countable state space give rise to standard p-functions; for the function p here a Markov chain representation is given in Remark 5.2 below.

Eq. (4.1) states that to evaluate the joint probability on the left-hand side it suffices to know the single probability function $p(t) = \Pr\{t \in \mathscr{U} \mid 0 \in \mathscr{U}\}$ for all $t > 0$. We call the PMS model described in the previous section with germs located at the points of a Poisson process with unit rate, the *standard PMS model*.

Theorem 5. *For the standard 1-D PMS model*,

$$p(t) = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\pi_n \quad (t \geqslant 0), \tag{4.2}$$

where

$$\pi_n = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \tag{4.3}$$

is the expected proportion of the circumference of a circle left uncovered by a PMS model generated by n germs located uniformly at random on the circumference.

Prelude to proof of Theorem 5. We can as easily describe a PMS model on a finite interval of the real line subject to the end-points acting as stopping the growth of the grains about the germs closest to each end of the interval. Let p_n denote the expected proportion of a finite interval not covered by the grains generated by the PMS model with n points placed at random on the interval.

Lemma 6. $\pi_n = ((n-1)/n)p_{n-2}$.

Proof. Let n points be located uniformly at random on a circle of unit circumference, and let Z_n denote the shortest distance between the n pairs of adjacent points; note that, because the minimum length of n i.i.d. exponential intervals with unit mean has mean $1/n$, and because the mean total length of the intervals is n, a simple conditioning argument gives $EZ_n = 1/n^2$. Grains grow around all n germs of the PMS model until for the first time two grains abut, at which stage all grains are of length Z_n. A proportion nZ_n of the circle is then covered; equivalently, $1 - nZ_n$ is uncovered. This uncovered portion contains $n-2$ germs, each embedded in a partially grown grain that has reached the size Z_n and does not touch any other grain. The proportion of the interval that remains uncovered in the remaining growth of the PMS model is the same as the proportion p_{n-2} of a finite interval that is uncovered after the growth of the PMS model when $n-2$ points are thrown at random on the interval and the end-points act as inhibitors of growth. Then, $\pi_n = E(1-nZ_n)p_{n-2}$. Lemma 6 is proved. □

Remark 4.1. The terms $\{\mathrm{e}^{-t}t^n/n!\}$ in (4.2) are those of a Poisson distribution with mean t, while $\pi_n \to \mathrm{e}^{-1}$ for $n \to \infty$, so $p(t) \to \mathrm{e}^{-1}$ $(t \to \infty)$, agreeing with Theorem 4(a). Combining (4.2) and (4.3) and interchanging summations and limits yield

$$\begin{aligned} p(t) &= 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{(-1)^j}{j!}\mathrm{e}^{-t}\frac{t^n}{n!} = 1 - \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\sum_{n=j}^{\infty}\mathrm{e}^{-t}\frac{t^n}{n!} \\ &= 1 - \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\int_0^t \mathrm{e}^{-u}\frac{u^{j-1}}{(j-1)!}\,\mathrm{d}u = 1 - \int_0^t \mathrm{e}^{-u}\sum_{j=0}^{\infty}\frac{(-u)^j}{j!(j+1)!}\,\mathrm{d}u \\ &= 1 - \int_0^t \mathrm{e}^{-u}\frac{J_1(2\sqrt{u})}{\sqrt{u}}\,\mathrm{d}u, \end{aligned} \tag{4.4}$$

where $J_1(x)$ is a Bessel function of the first kind and order 1. Since $J_1(x)$ is alternately positive and negative between its zeroes, of which there are infinitely many and some occur for arbitrarily large arguments, there are arbitrarily large t' and t'' for which $p(t') > \mathrm{e}^{-1} > p(t'')$ in spite of the convergence of $p(t)$ to its limit at an exponentially fast rate. The least value of $p(t)$ occurs at the first positive solution of $J_1(2\sqrt{t}) = 0$, namely $t = (3.83171/2)^2 = 3.6705$, with $p(3.6705) = 0.36622 = 1 - 0.63378$ (cf. Remark 3.2).

Remark 4.2. Throughout this paper, wherever interchange of limit operations occurs as for example in the derivation above, the necessary conditions for absolute convergence to justify the interchange are satisfied because the series are geometric or like those of an exponential series. No further comment will be made.

Remark 4.3. In the alternating renewal process interpretation of Ξ as covered and uncovered intervals, if we suppose that the mean lengths of these intervals are α and β, respectively, then $\beta = 1/\mu((0,\infty])$, $\alpha = \beta \int_{(0,\infty]} x\mu(\mathrm{d}x)$ and $\lim_{t\to\infty} p(t) = \beta/(\alpha+\beta) = 1/\mathrm{e}$, so $\alpha = \beta(\mathrm{e}-1)$. The uncovered intervals have the same exponential distribution as $X' \equiv X_0 - R_+$ conditional on $X' > 0$ (cf. (3.10) and (3.15)), and for $x > 0$

$$\Pr\{X' > x \mid X' > 0\} = \Pr\{X_0 > R_+ + x \mid X_0 > R_+\} = \mathrm{e}^{-x} E(\mathrm{e}^{-R_+})/E(\mathrm{e}^{-R_+}) = \mathrm{e}^{-x}.$$

Thus, $\beta = 1$ and, consistent with Section 2,

$$\alpha = \mathrm{e} - 1. \tag{4.5}$$

Proof of Theorem 5. To prove (4.2), observe that it can be written as

$$\begin{aligned}\frac{p(t)}{\mathrm{e}} &= \Pr\{\mathrm{O} \text{ and } t \in \mathscr{U}\}\\ &= \sum_{n=0}^{\infty} \Pr\{\mathrm{O} \text{ not covered by grains to left of O}\}\\ &\quad \times \Pr\{n \text{ germs lie in } (0,t) \text{ and none of the associated}\\ &\qquad\quad \text{grains touches O or } t\}\\ &\quad \times \Pr\{t \text{ not covered by grains to right of } t\}.\end{aligned} \tag{4.6}$$

By Theorem 4, each of the first and third terms here equals $1/\sqrt{\mathrm{e}}$. Also the probability that an arbitrarily chosen point on a circle of circumference t is not covered conditional on there being n points on the circle, is the same (by an unwrapping argument) as the middle term of (4.6) with n as there. Thus, (4.2) is established subject to proving (4.3). This we do by proving an equivalent formula for p_n and appealing to Lemma 6.

To determine p_n, choose a point u uniformly at random on the interval, yielding J germs to the left of u and $n-J$ to the right, where J follows the binomial distribution $\{b_j(u)\} = \{\binom{n}{j} u^j (1-u)^{n-j}\}$ (supposing, without loss of generality, that the interval is of unit length). Then for u to be uncovered, neither of the grains immediately to its left and right can cover it. Conditional on $J = j$, denote the probabilities of these events by $Q_{\mathrm{L},j}$ and $Q_{\mathrm{R},n-j}$. These events are independent, and relate to sets of j and $n-j$ uniformly distributed germs. Then $Q_{\mathrm{L},j} = Q_{\mathrm{R},j}$, and

$$p_n = \sum_{j=0}^{n} \int_0^1 b_j(u) Q_{\mathrm{R},j} Q_{\mathrm{R},n-j}\,\mathrm{d}u = \frac{1}{n+1} \sum_{j=0}^{n} Q_{\mathrm{R},j} Q_{\mathrm{R},n-j}. \tag{4.7}$$

Now n points distributed uniformly and independently on an interval, can be regarded as obtained from the first $n+1$ points of a Poisson process at unit rate, the last of these defining the end of the interval and the other n the points scattered on the interval as described. The PMS growth model occurs about these n points as germs, subject now to the constraint that the grain closest to the far end-point stops growing either when it touches that end-point or the grain on its other side, whichever occurs first; call this *end-point inhibition*. For this set-up, and much as at (3.10), define

$$q_n(y) = \Pr\{\text{O covered by time } y \text{ from growth model on } n \text{ germs with end-point inhibition}\}.$$

Then by the same argument as leads to (3.11),

$$q_n(y) = \int_0^y \mathrm{e}^{-x}\mathrm{e}^{-x}[1 - q_{n-1}(x)]\,\mathrm{d}x, \tag{4.8}$$

where $q_0(y) = 0$ (all $y > 0$) and $q_i(0+) = 0$ $(i = 1, 2, \ldots)$. Set $Q_z(y) = \sum_{n=1}^{\infty} z^n q_n(y) = \sum_{n=0}^{\infty} z^n q_n(y)$ for $|z| < 1$. Then multiplying (4.8) by z^n and summing on $n = 1, 2, \ldots,$ gives

$$Q_z(y) = \int_0^y \mathrm{e}^{-2x}\left[\frac{z}{1-z} - zQ_z(x)\right]\mathrm{d}x. \tag{4.9}$$

Solving this integral equation as for (3.11) leads to

$$\log[1 - (1 - z)Q_z(y)] = -\tfrac{1}{2}z(1 - \mathrm{e}^{-2y}). \tag{4.10}$$

Let $y \to \infty$, and note that $Q_{\mathrm{R},0} = 1$ and for $n = 1, 2, \ldots,$ $Q_{\mathrm{R},n} = 1 - q_n(\infty)$. Then, after rearrangement,

$$\sum_{n=0}^{\infty} z^n Q_{\mathrm{R},n} = \frac{\exp(-\frac{1}{2}z)}{1-z} = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \frac{(-\frac{1}{2})^j}{j!}, \tag{4.11}$$

hence a more detailed version of Theorem 4.

Lemma 7.

$$Q_{\mathrm{R},n} = \sum_{j=0}^{n} \frac{(-\frac{1}{2})^j}{j!} \quad (n = 0, 1, \ldots). \tag{4.12}$$

We can now use (4.7) to compute

$$p_n = \frac{1}{n+1}\sum_{i=0}^{n} Q_{R,i}Q_{R,n-i} = \frac{1}{n+1}\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{n-i} \frac{(-\frac{1}{2})^{j+k}}{j!k!}. \tag{4.13}$$

In this summation, the typical term as shown occurs for $n+1-(j+k)$ different values of i, so the sum of all terms for given $j + k = r$ say equals

$$(n + 1 - r)\sum_{j=0}^{r} \frac{(-\frac{1}{2})^r}{j!(r-j)!} = \frac{(n+1-r)(-1)^r}{r!}, \quad \text{using} \quad \sum_{j=0}^{r} \frac{r!}{j!(r-j)!} = 2^r.$$

Thus,

$$p_n = \sum_{i=0}^{n}\left(1 - \frac{i}{n+1}\right)\frac{(-1)^i}{i!}, \tag{4.14}$$

and therefore, using Lemma 6, for $n = 0, 1, \ldots,$

$$\pi_{n+2} = \frac{1}{n+2}\sum_{i=0}^{n}(n+1-i)\frac{(-1)^i}{i!} = \sum_{i=0}^{n+2}\frac{(-1)^i}{i!}, \tag{4.15}$$

after a modicum of algebraic manipulation. This last relation also agrees with $\pi_0 = 1$ and $\pi_1 = 0$, so (4.3) is established, and Theorem 5 is proved. □

Remark 4.4. The major part of this proof concerns the derivation of (4.3) for π_n. From that equation we see that π_n satisfies the recurrence relation, for $n=2,3,\ldots,$

$$n(\pi_n - \pi_{n-1}) = \frac{(-1)^n}{(n-1)!} = -(\pi_{n-1} - \pi_{n-2})$$

so $n\pi_n - (n-1)\pi_{n-1} = \pi_{n-2}$, or in terms of the p_j, using Lemma 6,

$$(n+1)p_n - np_{n-1} = n^{-1}(n-1)p_{n-2} \quad (n=2,3,\ldots), \tag{4.16}$$

with $p_0 = 1$, $p_1 = \frac{1}{2}$. Eq. (4.16) gives

$$n(n+1)p_n - n(n-1)p_{n-1} = np_{n-1} + (n-1)p_{n-2} \quad (n=2,3,\ldots),$$

from which we deduce that $\{p_n\}$ is determined by $p_0 = 1$, $p_1 = \frac{1}{2}$ and

$$n(n+1)p_n = np_{n-1} + \sum_{j=1}^{n-1} 2jp_{j-1} \quad (n=2,3,\ldots). \tag{4.17}$$

We can check (4.17) (and hence, by working backwards, derive (4.3)) as follows. Consider an interval of length x on which n germs are distributed uniformly at random, and let the growth process start at time zero with inhibition from the two end-points. Let p_n denote the proportion of the interval that ultimately remains uncovered. Consider the set-up after a small time interval h has passed, where $nh \ll x$. Neglecting terms that are $O(h^2)$ (recall that n is fixed), the expected length xp_n of the interval that is then uncovered satisfies

$$\begin{aligned} xp_n &= (x-2nh)p_n[1 - n^2 2h/x] + xp_{n-1}[n2h/x] \\ &\quad + \sum_{j=1}^{n-1} E(xZ_{j,n-1}p_{j-1} + x(1-Z_{j,n-1})p_{n-1-j})[n2h/x], \end{aligned} \tag{4.18}$$

where Z_{jm} is the jth order statistic in a sample of m from $(0,1)$ so $E(Z_{jm}) = j/(m+1)$, and the three terms on the right-hand side arise, respectively, from no touching of any of the grains growing about the n germs between themselves or of the end-points of the interval, from a growing grain touching an end-point, and from two grains touching one another in one of $n-1$ ordered configurations, with the terms $[\cdots]$ denoting the probabilities of these events. The term xp_n vanishes, and dividing the rest by $2h$ and taking the limit $h \downarrow 0$, leads to (4.17).

5. The covered intervals formed by contiguous grains

The implication of Remark 4.3 is that in the integral representation for the p-function $p(t)$ at (4.2), the Laplace–Stieltjes transform $\gamma(\cdot)$ say, of the distribution $G(\cdot)$ for the length covered by a set of contiguous grains is related to the Laplace transform of p as in

$$\int_0^\infty e^{-\theta t} p(t)\,dt = \frac{1}{\theta + 1 - \gamma(\theta)}. \tag{5.1}$$

We find the left-hand side of (5.1) by taking the transform of the double sum leading to (4.4): for $\mathrm{Re}(\theta) > 0$,

$$\int_0^\infty \mathrm{e}^{-\theta t} p(t)\,\mathrm{d}t = \frac{1}{\theta} + \sum_{n=1}^{\infty}\sum_{j=1}^{n} (1+\theta)^{-(n+1)} \frac{(-1)^j}{j!}$$

$$= \frac{1}{\theta}\left[1 + \sum_{j=1}^{\infty} \frac{1}{j!}\left(-\frac{1}{1+\theta}\right)^j\right] = \frac{1}{\theta}\exp\left[-\frac{1}{1+\theta}\right], \tag{5.2}$$

so

$$\theta + 1 - \gamma(\theta) = \theta \exp\left[\frac{1}{1+\theta}\right] = \sum_{j=0}^{\infty} \frac{\theta}{(1+\theta)^j}\frac{1}{j!}$$

$$= \theta + 1 - \sum_{j=1}^{\infty} \frac{1}{(1+\theta)^j}\left[\frac{1}{j!} - \frac{1}{(j+1)!}\right]. \tag{5.3}$$

The distribution function $G(\cdot)$ therefore has density g say given by

$$g(x) = \sum_{j=0}^{\infty}\left[\frac{1}{(j+1)!} - \frac{1}{(j+2)!}\right]\frac{\mathrm{e}^{-x}x^j}{j!} = \mathrm{e}^{-x}\left[\frac{I_1(2\sqrt{x})}{\sqrt{x}} - \frac{I_2(2\sqrt{x})}{x}\right], \tag{5.4}$$

where $I_k(\cdot)$ denotes a modified Bessel function of order k. Integrating the series in (5.4) on $(0,\infty)$ completes the proof of Theorem 8.

Theorem 8. *The length of a randomly chosen set of contiguous grains has d.f. G satisfying*

$$1 - G(x) = \mathrm{e}^{-x}\sum_{j=0}^{\infty}\frac{x^j}{j!(j+1)!} = \mathrm{e}^{-x}\frac{I_1(2\sqrt{x})}{\sqrt{x}}. \tag{5.5}$$

Eq. (5.5) enables us to check (4.5) directly: $\alpha = \int_0^\infty [1-G(x)]\,\mathrm{d}x = \sum_{i=0}^{\infty} 1/(i+1)! = \mathrm{e}-1$. We can as easily compute the second moment of the length of these intervals of contiguous grains by

$$\int_0^\infty 2x[1-G(x)]\,\mathrm{d}x = 2\sum_{i=0}^{\infty}\frac{1}{i!} = 2\mathrm{e},$$

so the variance of these lengths equals $4\mathrm{e} - \mathrm{e}^2 - 1 = 2.48407$, with coefficient of variation=0.84135.

5.1. Distribution of the number of grains in a covered interval

From the alternating renewal process interpretation of the successive covered and uncovered intervals of Ξ, it follows that the numbers of germs N_j in successive covered intervals to the right of a randomly chosen uncovered interval, are i.i.d. r.v.s. From ergodicity, supposing there are n_t distinct pairs of uncovered and covered intervals (of average combined length $\alpha + 1 = \mathrm{e}$) in a long segment of the real line of length t, in which there are (by ergodicity) $N_1 + \cdots + N_{n_t} = t + \mathrm{o}(t)$ germs, $E(N_1) = \lim_{t\to\infty}(t/n_t)$, and $n_t\times$ (average combined length) $\approx t$, so $E(N_1) = \mathrm{e}$. Write $\{\phi_k\} = \{\Pr\{N_j = k\}\}$ for the common distribution of the N_j.

Suppose instead that we choose a set of $k-1$ independent exponentially distributed intervals between k adjacent germs for some $k\geqslant 2$; let ϕ_k' denote the probability that they are the germs of a covered interval with k contiguous grains. Now left-most germs of covered intervals are in one–one correspondence with the intervals, so the probability that a randomly chosen germ is a left-most germ equals e^{-1}. Hence,

$$\phi_k' = \mathrm{e}^{-1}\phi_k. \tag{5.6}$$

To find $\{\phi_k\}$, we first examine more closely the qualitative relationships of contiguous grains. It is useful to call a pair of contiguous grains of the same radius a *doublet*, and an *isolated doublet* if neither of them touches any other grain.

Lemma 9. *If grains* G_i *of radii* R_i $(i = 1, 2, 3)$ *are such that* G_2 *is contiguous with the other two and* $R_1 \leqslant R_2$, *then* $R_2 < R_3$ *with probability one.*

Proof. We show that $\Pr\{R_2 \geqslant R_3\} = 0$. If $R_2 \geqslant R_3$ then at the epoch that G_2 stops growing it simultaneously touches the other two grains, which are either both growing (this is the case $R_1 = R_2 = R_3$), or one is growing and the other has already stopped growing (e.g. $R_1 = R_2 > R_3$), or both have stopped growing (so $R_1 < R_2 > R_3$). Denoting the distances between the germs by X_1 and X_2, the first case requires $X_1 = X_2$, the second $\frac{1}{2}X_1 = X_2 - R_3$, and the third, $X_1 - R_1 = X_2 - R_3$. The r.v.s X_1 and X_2 are independent and have absolutely continuous distributions, so $\Pr\{X_1 = X_2\} = 0$. Next, $R_3 < R_2$ means that grain G_3 stops growing before G_2 so that $X_2 > 2R_3$ and the radius R_3 is determined by P_3 and germs to its right, i.e. R_3 is determined independently of germs to the left of P_2. Then $X_2 - R_3$ is independent of X_1 and has an absolutely continuous distribution, so $\Pr\{X_2 - R_3 = \frac{1}{2}X_1\} = 0$. Similarly, $\Pr\{X_1 - R_1 = X_2 - R_3\} = 0$ in the third case. □

We give two simple consequences of this lemma (see Section 7 for d-D analogues).

Lemma 10. *With probability one*:
(a) *every covered interval of* Ξ *of contiguous grains, contains exactly one doublet*;
(b) *for a covered interval containing* N *contiguous grains, the radii of consecutive grains satisfy*

$$R_{-N''} > R_{-(N''-1)} > \cdots > R_{-1} = R_1 < \cdots < R_{N'} \tag{5.7}$$

for some positive integers N' *and* N'' *for which* $N' + N'' = N$.

A result that we do not use concerns intervals between germs satisfying $X_0 < X_1 > X_2$ when the central one of these three is covered by a doublet. Then the doublet is isolated.

We now work towards evaluating ϕ_k'. First, suppose an interval of length X_0 is covered by a doublet, of common radii $\frac{1}{2}X_0$. Much as in Section 3 for example, the configuration of germs and grains to the right and left of this interval, conditional on X_0, are independent and identically distributed. In particular then, conditional on $X_0 = x$, the numbers $N'(x) - 1$ and $N''(x) - 1$ of contiguous grains to the right and left of X_0, forming a covered interval containing $N'(x) + N''(x)$ germs in all, are i.i.d.

Suppose next that we choose a particular set of independent exponentially distributed intervals between k adjacent germs, and that they are the k germs of a covered interval comprised of k contiguous grains. We noted in Lemma 10 that any set of contiguous grains contains exactly one doublet; let X_0 denote the length of the interval between adjacent germs covered by this doublet. Conditional on $X_0 = x$, the numbers of germs in the covered interval on each side of X_0 are independent, so, recalling that a doublet interval has length x if and only if the pair of germs concerned are distance x apart,

$$\begin{aligned}\phi_k'(x) &\equiv \Pr\{k-1 \text{ intervals form a covered interval with } k \text{ germs} \mid \text{doublet} \\ &\qquad \text{interval has length } x\} \\ &= \sum_{j=1}^{k-1} p_j(x)\, p_{k-j}(x),\end{aligned} \tag{5.8}$$

where $p_j(x) = \Pr\{N'(x) = j\}$ and $N'(x)$ is the conditional r.v. discussed above. We evaluate $p_j(x)$ in Lemma 12 below. Substituting from there,

$$\phi_k'(x) = \sum_{j=1}^{k-1} \frac{e^{-\frac{1}{2}-(j-\frac{1}{2})x}(\frac{1}{2})^{j-1}}{(j-1)!} \frac{e^{-\frac{1}{2}-(k-j-\frac{1}{2})x}(\frac{1}{2})^{k-j-1}}{(k-j-1)!} = \frac{e^{-1-(k-1)x}}{(k-2)!}, \tag{5.9}$$

so $\phi_k' = \int_0^\infty \phi_k'(x) e^{-x}\,dx = e^{-1}(k-1)/k!$. Using (5.6) proves Theorem 11.

Theorem 11. *The number N of contiguous grains in a covered interval has the distribution*

$$\phi_k = \Pr\{N = k\} = \frac{k-1}{k!}, \quad (k = 2, 3, \ldots). \tag{5.10}$$

Lemma 12. $p_m(x) = e^{-\frac{1}{2}-(m-\frac{1}{2})x}(\frac{1}{2})^{m-1}/(m-1)!$.

Proof. Suppose as above that an interval of length X_0 is covered by a doublet whose grains have radii $R_0 = R_1 = \frac{1}{2}X_0$. Then the right-hand grain of this doublet is an end grain if the interval length X_1 to the next germ and the radius R_+ of the grain that would grow around that germ in the absence of the doublet and germs to its left, are such that $X_1 - R_1 > R_+$. Therefore,

$$\begin{aligned}\Pr\{N'(x) = 1 \mid X_0 = x\} &= \Pr\{X_1 - R_1 > R_+ \mid X_0 = x\} \\ &= E(e^{-R_1 - R_+} \mid X_0 = x) = e^{-\frac{1}{2}x - \frac{1}{2}},\end{aligned} \tag{5.11}$$

using the independence of R_+ of X_0, $R_1=\frac{1}{2}X_0$ and Eq. (3.15) from Remark 3.1. This proves Lemma 12 in the case $m=1$. We use these properties of R_+ again in the general case $m\geqslant 2$ to which we now turn.

Lemma 10 shows that for $N'(x)=m$ to hold the radii must satisfy $R_m > R_{m-1} > \cdots > R_1 = \frac{1}{2}x$ and the intervals of lengths $X_1,\dots,X_{m-1}$ between germs $\mathrm{P}_1,\mathrm{P}_2,\dots,\mathrm{P}_m$, must be covered but the next interval of length X_m must be partly uncovered. Thus, $\{N'(x)=m\}$ when for such m all three of the events below hold:

$$A_m(x)\equiv\{X_0=x, X_i=R_i+R_{i+1}\quad (i=1,\dots,m-1)\},$$

$$B_m(x)\equiv\{R_m>R_{m-1}>\cdots>r_1=\tfrac{1}{2}x=\tfrac{1}{2}X_0\},\tag{5.12}$$

$$C_m\equiv\{X_m>R_m+R_+\},$$

where R_i is the radius of the grain with centre P_i, X_i is the distance between germs P_i and P_{i+1}, and R_+ is the radius of the grain at P_{m+1} when determined purely by germs to the right of its centre and the germs P_i $(i\leqslant m)$ play no part. Then

$$\begin{aligned}p_m(x)&=E[I(A_m(x)\cap B_m(x)\cap C_m)]\\&=E[\Pr\{X_m>R_m+R_+\,|\,R_+,A_m(x)\cap B_m(x)\}I(A_m(x)\cap B_m(x))]\\&=E[\exp(-R_m-R_+)I(A_m(x)\cap B_m(x))]\\&=E(\mathrm{e}^{-R_+})E[\mathrm{e}^{-R_m}I(A_m(x)\cap B_m(x))]\\&=\mathrm{e}^{-\frac{1}{2}}\int\cdots\int_{A_m(x)\cap B_m(x)}\mathrm{e}^{-r_m}\mathrm{e}^{-x_1-\cdots-x_{m-1}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{m-1},\end{aligned}\tag{5.13}$$

where the r.v.s X_i and R_i in the sets $A_m(x)$ and $B_m(x)$ are replaced by the corresponding variables x_i and r_i, so the latter satisfy $r_{i+1}=x_i-r_i$ $(i=1,\dots,m-1)$ and $r_1=\frac{1}{2}x$. Define the mapping $(x,x_1,\dots,x_{m-1})\mapsto(x,z_1,\dots,z_{m-1})$ by $z_i=r_{i+1}-r_i=x_i-2r_i$ $(i=1,\dots,m-1)$ so that $r_{i+1}=z_i+\cdots+z_1+\frac{1}{2}x$ and $x_i=z_i+2r_i=z_i+2(z_{i-1}+\cdots+z_1)+x$. This mapping has unit Jacobian and the region $A_m(x)\cap B_m(x)$ is mapped into $\{z_i>0:\ i=1,\dots,m-1\}$. Then for the integrand at (5.13),

$$\begin{aligned}r_m+\sum_{i=1}^{m-1}x_i&=\tfrac{1}{2}x+\sum_{j=1}^{m-1}z_j+\sum_{i=1}^{m-1}\left[z_i+2\sum_{j=1}^{i-1}z_j+x\right]\\&=\left(m-\tfrac{1}{2}\right)x+\sum_{j=1}^{m-1}2(m-j)z_j,\end{aligned}\tag{5.14}$$

so

$$\begin{aligned}\mathrm{e}^{\frac{1}{2}}p_m(x)&=\int_0^\infty\mathrm{d}z_1\dots\int_0^\infty\mathrm{e}^{-(m-\frac{1}{2})x}\mathrm{e}^{-\sum_{j=1}^{m-1}2(m-j)z_j}\,\mathrm{d}z_{m-1}\\&=\mathrm{e}^{-(m-\frac{1}{2})x}\frac{(\frac{1}{2})^{m-1}}{(m-1)!}.\end{aligned}\tag{5.15}$$

Observe in passing that $p_{m+1}(x)=\frac{1}{2}\mathrm{e}^{-x}p_m(x)/m$.

The mean $E(N_j)=\sum_{k=2}^{\infty} k\phi_k = \mathrm{e}$ we have noted. Similarly, $\operatorname{var} N_j = (3-\mathrm{e})\mathrm{e} = 0.76579$. Also, from Theorem 11, a covered interval is an isolated doublet with probability $\frac{1}{2}$. Peter Winkler (of Bell Labs (Lucent)) asked about the 'collisions' that occur between grains in the growth process. Either both grains are 'live' and growing, or one is live and the other 'dead', having already stopped growing: what is f_{g}, the fraction of collisions involving two live grains? From Lemma 10, in a set of N contiguous grains, there is one doublet (resulting from a collision of two live grains), and the other $N-2$ collisions are from one live and one dead grain. Since $E(N-2)=\mathrm{e}-2$, $f_{\mathrm{g}} = 1/[1+(\mathrm{e}-2)] = 1/(\mathrm{e}-1)$.

5.2. The length of a covered interval and the number of contiguous grains in it

The proof of the next result uses a method akin to our original proofs of both Theorems 1 and 3.

Theorem 13. *The length W_k of a covered interval conditional on its containing k germs, has the same distribution as the sum of $k-1$ independent unit exponential r.v.s.*

Proof. It is enough to show that for $\mathrm{Re}(\theta)\geqslant 0$,

$$\psi_k(\theta) \equiv E[\mathrm{e}^{-\theta W_k} I(\text{covered interval contains } k \text{ germs})] = \frac{\phi_k}{(1+\theta)^{k-1}}. \tag{5.16}$$

Much as in the proof of Theorem 11, we can write $W_k = W'_m + W''_{k-m}$ where the r.v.s W'_m and W''_{k-m} are conditionally independent given the common length x of the grains of the doublet contained in the covered interval concerned. In terms of the notation introduced around (5.12), $W'_m = R_m + X_{m-1} + \cdots + X_1 + \frac{1}{2}X_0$, so we find

$$p_m(\theta \,|\, x) \equiv E[\mathrm{e}^{-\theta W'_m} I(A_m(x) \cap B_m(x) \cap C_m) \,|\, X_0 = x], \tag{5.17}$$

and then exploit

$$\psi_k(\theta) = \sum_{j=1}^{k-1} \int_0^\infty p_j(\theta \,|\, x) p_{k-j}(\theta \,|\, x)\, \mathrm{e}^{-x}\, \mathrm{d}x, \tag{5.18}$$

since the r.v. X_0 has a unit exponential distribution. Much as around (5.13),

$$\begin{aligned} p_m(\theta \,|\, x) &= E[E[\mathrm{e}^{-\theta W'_m} I(\{X_m > R_m + R_+\}) \,|\, A_m(x) \cap B_m(x)] \,|\, X_0 = x] \\ &= E[\mathrm{e}^{-\theta W'_m} \mathrm{e}^{-R_m - R_+} I(A_m(x) \cap B_m(x)) \,|\, X_0 = x] \\ &= \mathrm{e}^{-\frac{1}{2}} \int \cdots \int_{A_m(x)\cap B_m(x)} \mathrm{e}^{-\theta w'_m - r_m}\, \mathrm{e}^{-x_{m-1} - \cdots - x_1}\, \mathrm{d}x_1 \ldots \mathrm{d}x_{m-1}, \end{aligned}$$

in obvious notation for w'_m and $A_m(x)$ and $B_m(x)$ as before. The exponent of the integrand here, again using the relations $x_i = z_i + 2r_i$ and $r_{i+1} = z_i + r_i$, equals

$$-\theta m x - (m - \tfrac{1}{2})x - (\theta + 1)\sum_{j=1}^{m-1} 2(m-j) z_j,$$

so after transforming the integral and integrating over $0 < z_i < \infty$ for $i = 1, \dots, m-1$ we find

$$p_m(\theta|x) = e^{-\frac{1}{2}} e^{-(1+\theta)mx} e^{-\frac{1}{2}x} \frac{(\frac{1}{2})^{m-1}}{(1+\theta)^{m-1}(m-1)!}. \tag{5.19}$$

Substituting in (5.18) and dividing by the probability e^{-1} yields (5.16) as required. □

Remark 5.1. Combining Theorems 11 and 13 gives another proof of Theorem 8, without reference to the Kingman representation (5.1). To see this, use (5.10) and (5.16) to yield

$$\gamma(\theta) = E[\exp(-\theta(\text{length of covered interval}))] = \sum_{k=2}^{\infty} \phi_k E(e^{-\theta W_k})$$

$$= \sum_{k=2}^{\infty} \frac{k-1}{k!} \frac{1}{(1+\theta)^{k-1}},$$

which is the same as (5.3).

Remark 5.2. Theorem 13 has as an immediate corollary the following identification of a continuous time Markov chain $\{X(t) : t \geqslant 0\}$ on countable state space for which the standard p-function $p(t)$ at (4.2) is a diagonal transition probability function. Take the positive integers $\{1, 2, \dots\}$ as state space, and define the matrix $\mathbf{Q} = (q_{ij})$ of transition rates by

$$q_{ij} = \begin{cases} -1 & (j = i), \\ 1 & (j = i-1, \ i = 2, 3, \dots), \\ (j-1)/j! & (i = 1, \ j = 2, 3, \dots), \\ 0 & \text{otherwise}. \end{cases} \tag{5.20}$$

Then $p_{11}(t) \equiv \Pr\{X(t) = 1 \mid X(0) = 1\} = p(t)$.

6. Grain-length distributions

Simulations reported in [DSS] indicate that the volume V of a typical grain in the PMS model has a distribution that differs from the exponential in a broadly similar fashion no matter whether in dimension 1, 2 or 3. We have already identified the 1-D grain-length distribution in Theorem 3, namely $\Pr\{V > y\} = \exp(e^{-y} - y - 1)$. It is pertinent to note that using the inequalities $1 - y < e^{-y} < 1 - y + \frac{1}{2}y^2$, where $y > 0$, gives the bounds below, deduced by other d-D arguments in [DSS]:

$$e^{-2y} < e^{-y} \exp[-(1 - e^{-y})] < e^{-y} \left[1 - (1 - e^{-y}) + \tfrac{1}{2}(1 - e^{-y})^2\right]$$

$$= \tfrac{1}{2} e^{-y}(1 + e^{-2y}). \tag{6.1}$$

In the setting of Remark 3.1, the random variable R_+, conditional on $R_+ < X_0$, is the half-length of a typical grain located at either end of an interval covered by two or more contiguous grains as discussed in Section 5 above. Recall that each grain touches either one or two other grains, depending on whether it is an ‘end grain’ or not.

Theorem 14. *The length V_e of an end grain has distribution*

$$\Pr\{V_e \leqslant y\} = (1-e^{-y})\exp\left(\tfrac{1}{2}e^{-y}\right). \tag{6.2}$$

Proof. When O is uncovered, the grain around the first germ P_1 to its right, at distance X_0 say, is an end grain. The radius of this grain in the absence of any germs to the left of O, is represented by the generic r.v. R_+, irrespective of whether $R_+ <$ or $> X_0$. Consequently, the length V_e of an end grain satisfies

$$\Pr\left\{\tfrac{1}{2}V_e > x\right\} = \Pr\{R_+ > x \mid X_0 > R_+\} = \frac{\Pr\{X_0 > R_+ > x\}}{\Pr\{X_0 > R_+\}}, \tag{6.3}$$

with X_0 independent of R_+. The denominator equals $e^{-\frac{1}{2}}$ by (3.15), and the numerator equals

$$\int_x^\infty \Pr\{X_0 > y\}\, r(y)\,dy, \tag{6.4}$$

where $r(\cdot)$ is the probability density function of R_+. By (3.9),

$$r(x) = (1+e^{-2x})\exp\left(\tfrac{1}{2}e^{-2x} - x - \tfrac{1}{2}\right).$$

Thus,

$$\begin{aligned}\Pr\{V_e > 2x\} &= \frac{1}{e^{-\frac{1}{2}}}\int_x^\infty e^{-y}(1+e^{-2y})\exp\left(\tfrac{1}{2}e^{-2y} - y - \tfrac{1}{2}\right)\,dy\\ &= \frac{1}{2}\int_0^{e^{-2x}} (1+v)\exp\left(\tfrac{1}{2}v\right)\,dv\\ &= 1-(1-e^{-2x})\exp\left(\tfrac{1}{2}e^{-2x}\right).\end{aligned}$$

This is equivalent to (6.2).

The regenerative nature of $\mathscr{U}$ and conditional independence of germs on each side of any given point in $\mathscr{U}$, implies the following corollary.

Theorem 15. *A randomly chosen interval between germs that includes uncovered points, has length X' whose distribution is that of $X_0 + \frac{1}{2}(V_e' + V_e'')$, where these three r.v.s are mutually independent and V_e' and V_e'' have the same distribution as V_e.*

For moments we have

$$\begin{aligned}E(V_e) &= \int_0^\infty \left[1-(1-e^{-y})\exp\left(\tfrac{1}{2}e^{-y}\right)\right]dy = \int_0^{1/2}\left[2e^u - \frac{e^u - 1}{u}\right]du\\ &= 2(e^{\frac{1}{2}}-1) - \mathrm{Ei}(\tfrac{1}{2}) + \log\tfrac{1}{2} + \gamma = 0.727291,\end{aligned}$$

where $\mathrm{Ei}(\cdot)$ denotes the exponential integral function (see e.g. Abramowitz and Stegun, 1972, Eq. (5.1.40)) and γ denotes Euler's constant. Alternatively, using power series,

$$E(V_{\mathrm{e}}) = \int_0^\infty \sum_{k=1}^\infty \frac{(2k-1)(\frac{1}{2})^k \mathrm{e}^{-ky}}{k!}\,\mathrm{d}y = \sum_{k=1}^\infty \frac{(2k-1)(\frac{1}{2})^k \mathrm{e}^{-ky}}{k!\,k} = 0.727291,$$

$$E(V_{\mathrm{e}}^2) = \int_0^\infty 2y \sum_{k=1}^\infty \frac{(2k-1)(\frac{1}{2})^k \mathrm{e}^{-ky}}{k!}\,\mathrm{d}y$$

$$= 2\sum_{k=1}^\infty \frac{(2k-1)(\frac{1}{2})^k \mathrm{e}^{-ky}}{k!\,k^2} = 0.684176 + [E(V_{\mathrm{e}})]^2.$$

Hence, E(total length of grains at ends of intervals of contiguous grains) $=2\times 0.727291=$ 1.45458 and therefore

$$E(\text{average length of grains totally interior to contiguous intervals})$$

$$= \frac{\alpha - 1.45458}{\mathrm{e}-2} = 0.367125,$$

where the ratio here is of the average length of wholly interior grains, and the average number of interior grains (we have used an ergodic argument again). Observe that 'interior' grains are of rather smaller average length than 'end grains', as is consistent with the U-shaped nature of the sequence of the successive radii of contiguous grains noted in Lemma 10. Theorem 13 implies that the mean length of grains in a covered interval comprised of k contiguous grains, equals $1-k^{-1}$.

7. Concluding remarks

It is worth reflecting on the origins and results of this paper and contemplate possible further work. The model discussed is a particular case of Helga Stoyan's extension of the simpler germ–grain model of Stienen (1982) (cf. e.g. Stoyan et al. 1995, p.218); independently, Häggström and Meester (1996) described it and called it the dynamic lily-pond model. Two points arise:

(i) We have ruthlessly exploited the independence properties of points of a Poisson process in disjoint regions, specifically, the half-intervals to the left and right of any given point on the real line. The curse of dimensionality debars us from exploiting these properties in the same way for the d-D model for $d \geqslant 2$.

(ii) In the Stienen germ–grain model, germs are located at the points of a Poisson process but each grain, with centre P_i say and nearest distance of the neighbour d_i, is a sphere of radius $\frac{1}{2}d_i$. The volume fraction $\varpi_d = (\frac{1}{2})^d$ is therefore lower than for the PMS model, but is explicitly computable. The volume of a typical grain (cf. Theorem 3) is exponentially distributed with mean $v_d(\frac{1}{2})$ where $v_d(y)$ is the volume of a d-D sphere of diameter y.

Qualitative results like those in Lemmas 9 and 10 about contiguous grains have been studied in more detail for the general d-D model in Häggström and Meester (1996). They arise in their proof that in the d-D model there is with probability one no cluster of infinitely many contiguous grains (such a question is a basic one to be answered of any potential model for percolation).

Extending quantitative results to the d-D model is much more problematic, because the order of properties of the reals on a half-line are largely lost in any half-space with $d \geqslant 2$. Still, [DSS] gives simulation results for certain d-D properties which, after a suitable d-dependent transformation, are approximately independent of $d = 1, 2, 3$. There is then value in knowing their exact form for $d = 1$ (cf. comment preceding Eq. (6.1)). For example, because we know the distribution of the 1-D grain volume V, we can compare the relative accuracy of estimating ϖ_1 by direct simulation of n replicates of the indicator function $I(t)$ that a particular point is covered, so this estimate $\hat{\varpi}_1$ has $\operatorname{var}\hat{\varpi}_1 = (\operatorname{var} I(t))/n = \varpi_1(1-\varpi_1)/n = (\mathrm{e}-1)/n\,\mathrm{e}^2 = (0.4822)^2/n$, and by a marked point process as in [DSS] where when n grain volumes are simulated with mean $\bar{V}$, $\operatorname{var}(\tilde{\varpi}_1) = \operatorname{var}\bar{V} = (\operatorname{var} V)/n = (0.7550)^2/n$. From [DSS], a typical d-D volume V_d has V_d/ϖ_d approximately independent of d, so the analogous estimators have

$$\operatorname{var}\hat{\varpi}_d = \frac{\varpi_d(1-\varpi_d)}{n}, \quad \operatorname{var}\tilde{\varpi}_d = \frac{\operatorname{var} V_d}{n} \approx \frac{\operatorname{var}(\varpi_d V_1/\varpi_1)}{n} = \frac{\varpi_d^2(\operatorname{var} V_1)/\varpi_1^2}{n}, \tag{7.1}$$

where $\varpi_d(\operatorname{var} V_1)/\varpi_1^2 \to 0$ with increasing d (on the basis of simulation results in [DSS], it equals 0.90185, 0.498 and 0.265 for $d = 1,2,3$).

Schlather and Stoyan (1997) derive an integral formula for the second product moment

$$C_d(t) = \Pr\{2 \text{ points distance } t \text{ apart are both covered by grains}\}, \tag{7.2}$$

for the d-D Stienen model, deducing in particular that

$$C_1(t) = \tfrac{1}{4} + \tfrac{1}{2}\mathrm{e}^{-2t} - \tfrac{2}{5}\mathrm{e}^{-3t} + \tfrac{1}{4}\left(\tfrac{1}{3}t+1\right)\mathrm{e}^{-4t} + 2\mathrm{e}^{-3t/2} + \tfrac{1}{4}\left(t - 8\tfrac{2}{5}\right)\mathrm{e}^{-4t/3}. \tag{7.3}$$

The analogous function for the 1-D PMS model is available in terms of the function $p(t)$ in Theorem 3, because

$$\begin{aligned} C(t) &= \Pr\{\text{two points distance } t \text{ apart are both covered by } \Xi\} \\ &= 1 - \mathrm{e}^{-1} - \Pr\{\text{of two points distance } t \text{ apart, one is covered} \\ &\qquad\qquad \text{and the other not covered}\} \\ &= 1 - 2\mathrm{e}^{-1} + \Pr\{\text{two points distance } t \text{ apart are both uncovered}\} \\ &= 1 - 2\mathrm{e}^{-1} + \mathrm{e}^{-1}p(t). \end{aligned}$$

This function $C(t)$ is just the so-called (non-centred) covariance function (Stoyan et al., 1995, p. 202) or second product moment of the stationary random set formed by the union of the grains. The analogous correlation function $\rho(t) \equiv \operatorname{corr}(I(0), I(t))$ of the indicator process $I(t)$ that t is covered, is then given by

$$\rho(t) = \frac{C(t) - (1-\mathrm{e}^{-1})^2}{\mathrm{e}^{-1}(1-\mathrm{e}^{-1})} = \frac{\mathrm{e}p(t) - 1}{\mathrm{e} - 1}. \tag{7.4}$$

It follows that the function $\mathrm{e}p(t) - 1$, being a covariance, is positive definite on $t > 0$, and so has a Fourier representation (this fact for p-functions is given in Kingman, 1964). Comparison of $\rho(t)$ with the correlation function for the Stienen model based on $C_1(t)$ shows that both correlation functions are negative for some positive t but that $\rho(t)$ as at (7.4) oscillates infinitely often whereas the Stienen model correlation function $\downarrow 0$, monotonically, for $t \to \infty$.

Conceptually, the derivation of $C_1(t)$ for the Stienen model is not as elegant as for the PMS model; Schlather and Stoyan exploit a marked point process approach. If we were to attempt to repeat the approach of this paper, we could use the *covering* of an arbitrary point: O is covered if and only if either (i) the germs on each side, independent of the configuration of germs on the other, give rise to grains closest to O that cover O, or (ii) the germs on one side, independent of the germs on the other, gives a grain that covers O, and the germs on the other side neither yield a grain that covers O nor do they inhibit the coverage from the grain about a germ on the other side from covering O. We omit further details.

Acknowledgements

Work of LAS done in part with the support of a Visiting Fellowship from SMS, ANU.

References

Abramowitz, M., Stegun, A. (Eds.), 1972. Handbook of Mathematical Functions. Dover, New York.

Daley, D.J., Stoyan, D., Stoyan, H., 1999. The volume fraction of a Poisson germ model with maximally non-overlapping spherical grains. Adv. Appl. Probab. 31, 610–624.

Häggström, O., Meester, R., 1996. Nearest neighbor and hard sphere models in continuum percolation. Random Structural Algorithms 9, 295–315.

Kingman, J.F.C., 1964. The stochastic theory of regenerative events. Z. Wahrs. 2, 180–224.

Kingman, J.F.C., 1972. Regenerative Phenomena. Wiley, London.

Schlather, M., Stoyan, D., 1997. The covariance of the Stienen model. In: Jeulin, D. (Ed.), Advances in Theory and Applications of Random Sets. World Scientific, Singapore, pp. 157–174.

Stienen, J., 1982. Die Vergroeberung von Karbiden in reinen Eisen-Kohlenstoff-Staehlen. Dissertation, RWTH Aachen.

Stoyan, D., Kendall, W.S., Mecke, J., 1995. Stochastic Geometry and its Applications, 2nd Edition. Wiley, Chichester, UK.